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M.A./M.Sc. (Second Semester) EXAMINATION, May - June, 2022

MATHEMATICS

Paper First

(Advanced Abstract Algebra - II)

Time : Three Hours] *[Maximum Marks:80*

Note: Attempt all the sections as directed.

(Section - A)

(Objective/Multiple Choice Questions)

(each 1 mark)

Note: Attempt all questions. Choose the correct answer out of four alternative answers (A) through (D).

1. Which of the following is correct?
   (A) Every ring R is an R - module over itself
   (B) If U is ideal of R, then U is an R - module
   (C) Every abelian group is a module over the ring of integers.
   (D) All of these

P.T.O.

2. If the ring R has a unit element 1 and $1 \cdot a = a$ for all $a \in m$ then M is called:
   (A) A unital R - Module
   (B) Right R - Module
   (C) Left R - Module
   (D) None of these
3. If M is any R - Module, then M and {0} are always submodules of M. These are called_______submodules of M.
   (A) Proper
   (B) Improper
   (C) Subproper
   (D) Irreducible
4. Let T:M$\rightarrow$N be an R - Homomorphism. If B is a submodule of N, then:
   (A) T^{-1} (B) is submodule of N
   (B) T^{-1} (B) is submodule of M
   (C) T^{-1} (B) is kernel of R - homomorphism
   (D) T^{-1} (B) = T (M)
5. Pick the odd one out.
   (A) λ is an Eigen value of **A**
   (B) λ is a solution of the characteristic equation det $(\lambda I - A) = 0$
   (C) The system of equations $(\lambda I - A) = 0$ has trivial solutions.
   (D) There is a non - zero vector x such that $Ax - \lambda x$

6. Let T:V $\rightarrow$ V be a linear Operator and $T(x)=\lambda x$ for some scalar λ then x is called:

(A) An Eigenvector of T

(B) An Eigenvalue of T

(C) An Eigenspace of T

(D) None of these

7. Suppose that the characteristic polynomial of some matrix A is found to be $p(\lambda)=(\lambda-1)(\lambda-3)^2(\lambda-4)^2$. What is the size of A

(A) 5 × 5

(B) 6 × 6

(C) 5 × 6

(D) 6 × 5

8. What is the canonical form of the matrix $A=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(A) $x+xy+y^2$

(B) x^2+xy

(C) x^2+y^2

(D) x^2+xy+y^2

9. Zero subspace (0) and V are invariant subspace of V because:

(A) $OT \neq O$

(B) $OT = O \ and \ VT = V$

(C) $OT \neq O and VT = V$

(D) $OT = O \ and \ VT = V$

10. A Linear Transformation $T \in A_F(V)$ is called nilpotent is

(A) $T = O$

(B) $T^n \neq O$

(C) $T^n = O$

(D) $T \neq O$

11. Let $T \in A(V)$ be nilpotent. Then the subspace M of V, of dimension m, which is invariant under T, is called cyclic with respect to T if:

(A) $MT^m \neq \{0\}, MT^{m-1} \neq \{0\}$

(B) $MT^m \neq \{0\}, MT^{m-1} = \{0\}$

(C) $MT^m = \{0\}, MT^{m-1} \neq \{0\}$

(D) None of these

12. Smith normal form of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix}$

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

13. Rank of matrices $\begin{bmatrix} -2 & 0 & 10 \\ 0 & -3 & -4 \\ 1 & 2 & -1 \end{bmatrix}$

(A) Rank = 4

(B) Rank = 2

(C) Rank = 1

(D) Rank = 3

14. Let A be an mxn matrix over a PID R. Then the submodule of R^n generated by the m rows of A is called.

(A) Row Module

(B) Column Matrix

(C) Row Matrix

(D) Column Module

15. Let A be an mxn matrix over a PID. The common value of the row rank of A and the column rank of A is called:

(A) Invariant of A

(B) Rank of A

(C) Smith Normal form

(D) None of these

16. An element x of an R - module M is called _______ if $Ann(x) \neq \{0\}$; that is there exists a non - zero element $r \in R$ such that $rx = 0$.

(A) Torsion - free Element

(B) Torsion Module

(C) Torsion Element

(D) Torsion - Free Module

17. Find Invariant factor of $\begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}$ is

(A) $1, (x-2), (x-2)^2$

(B) $(x-2), (x-2)^3$

(C) $1, (x-2)^2$

(D) None of these

18. PIR means:

(A) Principal Right Ideal

(B) Principal Ideal Ring

(C) Principal Ideal Domain

(D) None of these

19. PID Means:

(A) Principal Right Ideal

(B) Principal Ideal Ring

(C) Principal Ideal domain

(D) None of these

20. First Isomorphism Theorem:

(A) $\frac{M}{\ker f} \overset{\infty}{\neq} \operatorname{Im} f$

(B) $Kerf \overset{\infty}{=} \operatorname{Im} f$

(C) $Kerf \overset{\infty}{=} N$

(D) None of these

Section - B

(Very Short Answer Type Questions)

(2 marks each)

Note: Attempt all questions. Answer in 2 - 3 sentences.

1. Define finitely Generated R - module and cyclic module.
2. Define Homomorphism of modules.
3. Define Index of nilpotency.
4. State primary decomposition Theorem.
5. Is B = $\begin{bmatrix} 1 & 3 & -2 \\ 1 & 3 & -2 \\ 1 & 3 & -2 \end{bmatrix}$ nilpotent? If yes, what is its index?
6. Define Noetherian and Artinian Ring.

7. Define Rational canonical form
8. State Generalized Jordan canonical form.

Section - C

(Short Answer Type Questions)

(3 marks each)

Note Attempt all questions.

1. Let R be commutative ring with unity and let $e \neq 0,1$ be an idempotent. Prove that Re cannot be a free R - module.
2. An R - module M is noetherian if and only if every submodule of M is finitely generated.
3. Show that a Jordan block J may be written as the sum of a scalar matrix and a nilpotent.
4. What is the characteristic polynomial F(t) and minimal polynomial P(t) of the following Jordan block A of order 4

$$A = \begin{pmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$ What are the characteristic roots of A?

5. Obtain smith normal form & rank for the matrix with integral enmies $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix}$
6. Let R be a principal ideal domain, and let M be an R - module. Then $TorM = \{x \in M \mid x$ is torsion$\}$ is a submodule OPM.

7. Find out the rational canonical form of the matrix whose invariant factors are $(x-3),(x-3)(x-1),(x-3)(x-1)^2$
8. Explain Generalized Jordan Form over any field.

Section - D

(Long Answer Type Questions)

(5 marks each)

Note: Attempt all questions.

1. Let M be a free R - module with a basis $\{e_1, e_2, -, e_n\}$ Then $M \overset{\infty}{=} R^n$.

OR

State and prove Wedderburn Artin Theorem.

2. Let $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda k \in F$ be distinct characteristic roots of $T \in A(V)$ and $V_1, V_2, \ldots\ldots, V_k$ be characteristic vectors of T belonging to $\lambda_1, \lambda_2, \ldots \lambda k$ resp. Then $V_1, V_2, \ldots\ldots, V_k$ are linearly independent over F.

OR

Let $T \in A(V)$. Then the characteristic and minimal polynomial for T have the same roots.

3. Find the Jordan Canonical form of A:

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

[11]

OR

State and prove fundamental theorem on Nilpotent.

4. Find the Invariant factors of the matrix

$$\begin{bmatrix} -x & 4 & -2 \\ -3 & 8-x & 3 \\ 4 & -8 & -2-x \end{bmatrix}$$ over the ring Q [x]. Also find the rank.

OR

Find Invariant factors, elementary divisors and Jordan canonical form of the matrix.

$$\begin{bmatrix} 5 & \frac{1}{2} & -2 & 4 \\ 0 & 5 & 4 & 4 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$